Research Program in Game Theory and Mathematical Economics

Research Memorandum No. 20

July 1966

AN APPLICATION OF NON-STANDARD ANALYSIS TO GAME THEORY

by

Eugene Wesley

Department of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel

The research described in this paper was partially supported by the United States Office of Naval Research, under Contract No. N62558-4355, Task No. NR 047-045. Reproduction in whole or in part is permitted for any purpose of the United States Government

1. Introduction

In this paper I shall present an application of an extended field of real numbers to the proof of a theorem in the theory of cooperative games. The proofs set forth below, which involve the use of A. Robinson's theory of non-standard analysis and are metamathematical in character, are not the only way in which the theorems can be verified; alternative proofs utilizing ordinary topological methods can in fact be carried out quite briefly. However, the attempt to apply non-standard analysis to game theory is novel. For this reason, what I have to show may be of interest, not only insofar as it presents new information on the theory of the kernel of a cooperative game, but also in that it demonstrates the possibility of effectively exploiting non-standard analysis as a tool for future investigation in this area. It could very well turn out, for example, that non-standard analysis could serve as a means by which concepts defined for games with a finite number of players could be extended to games with a continuum of players.

2. Definitions and Basic Concepts

N is a (finite or denumerably infinite) set of consecutive natural numbers, called players. v, the characteristic function, is a non-negative real function defined on the subsets of N, called coalitions, which satisfies

(2.1) $\quad v(\emptyset) = 0, \quad v(\{i\}) = 0$, for all i in N.

A <u>game</u> is a pair $(N;v)$. A coalition structure (C.S.) is a partition of N. An individually rational payoff configuration (i.r.p.c.) is a pair $(x;\mathcal{B})$, where $\mathcal{B}$ is a coalition structure and x is a real vector having one component for each member of N and satisfies: $x_i \geq 0$ for all i in N and $\Sigma_{i\in B} x_i = v(B)$ for all $B \in \mathcal{B}$. Let $(x;\mathcal{B})$ be an i.r.p.c. For all $S \subset N$ we denote

$$(2.2) \qquad e(S;x) = v(S) - \Sigma_{i \in S} x_i .$$

$e(S,x)$ is called the <u>excess</u> of S with respect to $(x;\mathcal{B})$. Further, let $i,j \in B \in \mathcal{B}$ and $i \neq j$; we denote

$$(2.3) \qquad \mathcal{T}_{ij} = \{S;\ S \subset N,\ i \in S,\ j \notin S\}$$

$$(2.4) \qquad S_{ij}(x) = \sup_{S \in \mathcal{T}_{ij}} e(S,x)$$

$$(2.5) \qquad \sigma(j,S) = v(S) - v(S - \{j\})$$

and

$$(2.6) \qquad \Omega(j) = \sup_{S \text{ a coalition}} \sigma(j,S)$$

We say that i <u>outweighs</u> j with respect to $(x;\mathcal{B})$ if $S_{ij}(x) > S_{ji}(x)$ and $x_j > 0$. The i.r.p.c. $(x;\mathcal{B})$ is <u>balanced</u> if there exists no pair of players h and k such that $h,k \in B \in \mathcal{B}$ and h outweighs k. The kernel $K(G)$ of a game G is the set of all balanced i.r.p.c.'s. The following theorem is known (see [2]; see also [1] and [3]):

<u>Theorem 2.1</u>. For any finite game G (a game consisting of a finite number of players) and any coalition structure $\mathscr{D}$ there exists a payoff vector x such that $(x;\mathscr{D})$ is in the kernel.

This theorem is in general untrue for infinite games.

<u>Example</u>: Consider the game $G = (N;v)$ where $N = \{1,2,3,\ldots\}$ and v is defined as follows:

$$(2.7) \qquad v(A) = \begin{cases} 1 & \text{For } A \text{ of the form } \{n, n+1, n+2,\ldots\} \\ 0 & \text{otherwise} \end{cases}$$

Choose the coalition structure $\mathscr{D} = \{N\}$. For this coalition structure there exists no payoff vector x for which $(x;\mathscr{D})$ is in the kernel of G.

<u>Proof</u>: By way of contradiction. Suppose that for some x, $(x;\mathscr{D})$ is in the kernel. If $x_n > 0$, then the coalition

$$C_{n+1} = \{n+1, n+2, \ldots\} \text{ is in } \mathscr{T}_{n+1,n}.$$

$$e(C_{n+1},x) = 1 - (x_{n+1} + x_{n+2} + \ldots) > 0.$$

On the other hand, for any coalition C in $\mathscr{T}_{n,n+1}$, $v(C) = 0$ and hence $e(C,x) = v(C) - \Sigma_{j\in C}x_j \leq -x_n < 0$. Thus $S_{n,n+1}(x) \leq -x_n < 0$. It follows therefore that

$$S_{n,n+1}(x) < e(C_{n+1},x) \leq S_{n+1,n}(x) \text{ and } x_n > 0.$$

This implies that n outweighs $n+1$, in contradiction to the hypothesis that $(x;\mathscr{D})$ is balanced. Thus $x_n = 0$ for all n. Therefore $x = (0,0,\ldots)$ which is impossible because Σx_i must equal $v(N)$ which is equal to 1. We thus see that the hypothesis that such an x exists leads to a contradiction.

<u>Definition 2.2</u>. $G = (N;v)$ is a superadditive game if for any two disjoint subsets C,D of N, $v(C \cup D) \geq v(C) + v(D)$.

3. The Non-Standard Model of a Game

We shall start with a brief definition and description of a non-standard model of analysis. For more complete details and for proofs the reader is referred to the first thirty pages of [4] or to the material appearing in the chapter on non-standard analysis in [5].

We begin by classifying real numbers and certain sets and relations into categories called types. We perform this classification inductively. A real number will be said to be of type 0. Suppose $A_1, \ldots, A_n$ are sets such that for every i, $1 \leq i \leq n$, A_i consists of elements all of which have been previously classified (by induction) into type t_i. Then any subset of $A_1 \times \ldots \times A_n$ will be said to be of type $(t_1, \ldots, t_n)$. Thus, 5 is of type 0. The set of all even numbers is of type (0). The order relation $<$, by set theoretic definition, is of type (0,0). Note: There exist elements that are of more than one type; the empty set, for example. The function $\cos zy$ may be said to be of type (0,0,0). The function may likewise be said to be of type ((0,0),0) or of type (0,(0,0)). We will now inductively define the length of a type. The type 0 will be said to be of length 1. If $t_1, t_2, \ldots, t_n$ have been previously (inductively) defined to be of lengths $\ell_1, \ell_2, \ldots, \ell_n$, then the type $(t_1, \ldots, t_n)$ will be said to be of length $\ell_1 + \ell_2 + \ldots + \ell_n + 2$. Let $\mathcal{L}_{30}$ be the set of all types of lengths less than 30. It is clear that $\mathcal{L}_{30}$ is a finite set.

Let A be the set of all elements that belong to at least one of the types in $\mathcal{L}_{30}$. Then A includes, among other things, all real numbers, all subsets of the set of real numbers, all subsets of $X \times X$, where X is the set of real numbers, and hence, by set theoretic definition of function, all functions of a single real variable.

Since a vector (finite or denumerably infinite) is essentially a real function defined on a subset of the set of natural numbers, A also contains all vectors. Similarly, it contains all measure functions defined on sets of real numbers. Thus it contains Sup and Σ.

Let $\mathcal{A} = \langle A; \emptyset_i, T_t, \emptyset_{ID}\rangle$ i-a natural number such that $i \geq 2$
t-a type

be a relational system, consisting of a set of individuals, and of a set of relations defined on the set of individuals. A, the set described in the preceding paragraph, is the set of individuals. The relations $\emptyset_i$, T_t and $\emptyset_{ID}$ are defined as follows:

$\emptyset_i$ is an i-ary relation on A. The i-ad $\langle a_1,\ldots,a_n\rangle$ (where $a_1,\ldots,a_n$ are elements in A) is said to be in $\emptyset_i$ if and only if a_1 is a set and the i-minus-1-ad $\langle a_2,\ldots,a_n\rangle$ is a member of a_1. For any type t, T_t is a one place relation on A. $b \in A$ is in T_t if and only if b is of type t. $\emptyset_{ID}$ is the binary identity relation on A.

Let L be a language made up of a set of symbols whose cardinality is greater than the cardinality of A, and of a one-to-one correspondence f from the elements of $\mathcal{A}$ (individuals and re-

lations) <u>into</u> L. Symbols that thus correspond to relations will be called predicates. Denote by K the set of all sentences in the calculus of predicates of first order formed from symbols of L that are meaningful in $\mathcal{A}$. Denote by K_0 the set of all sentences in K that are <u>true</u> in $\mathcal{A}$. Consider the following set of sentences.

(3.1) $K_1 = K_0 \cup \{\bar{Q}^1_{T,0}\,\bar{a}\} \cup \{\sim\bar{Q}^1_{ID}\,\bar{a}\bar{a}_\mu\}_\mu$ μ an index that runs through the real numbers.

Here $\bar{a}$ is a symbol in L that does not correspond under f to any element in $\mathcal{A}$. $\bar{Q}^1_{T,0}$ is the symbol in L that signifies (under f) the relation T_0. $\bar{Q}^1_{ID}$ is the symbol in L signifying $\emptyset_{ID}$. $\bar{a}_\mu$ is the symbol in L that corresponds to the number μ.

Since every finite subset of K_1 possesses a model ($\mathcal{A}$ is a model of every finite subset of K_1 (see [4], p. 18)), then by the compactness principle (ibid) K_1 itself possesses a model. Every model of K_1 shall be called a non-standard model of analysis.

Let $\mathcal{B}$ be some model of K_1. Every sentence in the predicate calculus of first order that is true when interpreted in $\mathcal{A}$ remains true when it is re-interpreted in $\mathcal{B}$. Numbers, sets, and relations in $\mathcal{A}$ are signified by symbols in L. These symbols, i[illegible], when re-interpreted in $\mathcal{B}$, signify certain elements in $\mathcal{B}$. Any such element will be called a $\mathcal{B}$-number, $\mathcal{B}$-set, or $\mathcal{B}$-relation, depending on whether the element in $\mathcal{A}$ signified by the corresponding symbol is a num-

ber, set or relation. For any element $\hat{c}$ in $\mathcal{B}$ which corresponds to a symbol $\bar{c}$ signifying some number c in $\mathcal{A}$, $\bar{Q}^1_{T,c}\bar{c}$ is true in $\mathcal{B}$. For any x in $\mathcal{B}$ such that $\bar{Q}^1_{T,c}x$, x will be called a $\mathcal{B}$-number. All other individuals in $\mathcal{B}$ are called $\mathcal{B}$-sets. The order relation $<$ in $\mathcal{A}$ carries over to a complete order relation on $\mathcal{B}$-numbers. The three place relation $+$ in A (a,b,c is in the relation if and only if $a+b = c$) passes over to a three place relation in $\mathcal{B}$ on $\mathcal{B}$-numbers. The number 0 passes over to σ in $\mathcal{B}$. There exist numbers in $\mathcal{B}$ greater than σ that are less than all $\mathcal{B}$-numbers signified by symbols corresponding to positive numbers in $\mathcal{A}$ (see [4]). Such $\mathcal{B}$-numbers are called infinitesimal. Infinite numbers are defined analogously. There exist $\mathcal{B}$-numbers and $\mathcal{B}$-sets not signified by any symbols signifying elements in $\mathcal{A}$. There exist sets whose elements all appear in $\mathcal{B}$ while the sets themselves do not appear in $\mathcal{B}$. Such sets are not $\mathcal{B}$-sets. $\mathcal{B}$-sets have properties that are analogous to those of $\mathcal{A}$-sets. They obey all axioms of set theory expressible in the predicate calculus of first order. We can thus speak of elements that are contained in a $\mathcal{B}$-set, intersections of $\mathcal{B}$-sets, $\mathcal{B}$-subsets of $\mathcal{B}$-sets, etc. Hence we can define $\mathcal{B}$-vectors, $\mathcal{B}$-functions and $\mathcal{B}$-relations in complete analogy with the set theoretic definitions of $\mathcal{A}$-vectors, $\mathcal{A}$-functions and $\mathcal{A}$-relations. We simply substitute the words $\mathcal{B}$-set for $\mathcal{A}$-set in each of the corresponding definitions. Denote by $\mathcal{N}$ the set of all natural numbers in $\mathcal{A}$.

Denote by $\hat{\mathcal{N}}$ the image of $\mathcal{N}$ in $\mathcal{B}$, i.e., the element in $\mathcal{B}$ signified by the symbol corresponding to $\mathcal{N}$. $\hat{\mathcal{N}}$ is a $\mathcal{B}$-set. Any $\mathcal{B}$-number contained in $\hat{\mathcal{N}}$ will be called a natural $\mathcal{B}$-number. There exist infinite as well as finite natural $\mathcal{B}$-numbers. $\mathcal{B}$-numbers for which there exist symbols corresponding to numbers in $\mathcal{A}$ will be called standard numbers. For any finite $\mathcal{B}$-number $\hat{d}$ there exists a unique standard number $\hat{d}^1$ such that $\hat{d}^1$ is the nearest standard number to $\hat{d}$ (see [4]). For any number e in $\mathcal{A}$ we shall denote the image of e in $\mathcal{B}$ by $e^{\sim}$. For any finite number $\hat{h}$ in $\mathcal{B}$ we shall denote the nearest standard number by $\hat{h}^{*}$. For any standard $\mathcal{B}$-number $\hat{\iota}$ we shall denote by $\hat{\iota}^{\vee}$ the image of $\hat{\iota}$ in $\mathcal{A}$. $\mathcal{B}$-elements will in general be denoted by lower case latin letters crowned by roofs ($\hat{b}$, $\hat{c}$, $\hat{d}$, etc.). $\mathcal{A}$-elements will be denoted in general by lower case uncrowned latin letters (p, q, r, etc.). $\hat{\emptyset}_{ID}$, the image of $\emptyset_{ID}$ in $\mathcal{B}$, may be assumed, without loss of generality, to be the identity relation. That is, if $\hat{a}$ and $\hat{b}$ are individuals in $\mathcal{B}$, the pair $<\hat{a},\hat{b}>$ is in $\hat{\emptyset}_{ID}$ if and only if $\hat{a}$ and $\hat{b}$ are both the same element.

We define a non-standard game in complete analogy with the standard $\mathcal{A}$-game given above. Let $\hat{N}$ be a $\mathcal{B}$-set of consecutive natural $\mathcal{B}$-numbers. If every number in $\hat{N}$ is less than some $\mathcal{B}$-number $\hat{c}$ then we say that $\hat{N}$ is $\mathcal{B}$-finite. Note: $\hat{N}$ may consist of an infinite number of $\mathcal{B}$-numbers and still be $\mathcal{B}$-finite. Let $\hat{v}$ be a $\mathcal{B}$-function defined on all $\mathcal{B}$-subsets of $\hat{N}$, whose values are non-negative $\mathcal{B}$-numbers; $\hat{v}(\emptyset) = 0^{\sim}$, $\hat{v}(\{\hat{i}\}) = 0^{\sim}$ for each $\hat{i}$ in $\hat{N}$. The pair $(\hat{N};\hat{v})$ is a non-standard game, or a $\mathcal{B}$-game. Let $\hat{\mathcal{D}}$ be a $\mathcal{B}$-set of $\mathcal{B}$-

subsets of $\hat{N}$ such that any two such subsets are disjoint and such that the union of the $\mathcal{B}$-subsets in $\hat{\mathcal{D}}$ is equal to $\hat{N}$. $\hat{\mathcal{D}}$ is then called a $\mathcal{B}$-coalition structure. Let $\hat{x}$ be a $\mathcal{B}$-vector having one coordinate for each element in $\hat{N}$. The pair $(\hat{x};\hat{\mathcal{D}})$ will be said to be a $\mathcal{B}$-i.r.p.c. if each coordinate of x is non-negative and $\Sigma_{\hat{i}\in\hat{D}}\hat{x}_{\hat{i}} = \hat{v}(\hat{D})$ for each $\hat{D}$ in $\hat{\mathcal{D}}$. The definitions of $\hat{e}(\hat{S};\hat{x})$, $\hat{\mathcal{J}}_{\hat{i},\hat{j}}$, $S_{\hat{i},\hat{j}}$, $\hat{\partial}(\hat{j},\hat{S})$ and $\hat{\Omega}(\hat{j})$ are entirely analagous to the definitions (2.2) - (2.6). The definition of balanced $\mathcal{B}$-i.r.p.c.'s in a $\mathcal{B}$-game is also completely analogous. The $\mathcal{B}$-kernel is the set of all $\mathcal{B}$-i.r.p.c.'s that are balanced. A $\mathcal{B}$-game is $\mathcal{B}$-finite if $\hat{N}$ is $\mathcal{B}$-finite.

<u>Lemma 3.1</u>. For any $\mathcal{B}$-finite game $\hat{G} = (\hat{N};\hat{v})$, and for any $\mathcal{B}$-coalition structure $\hat{\mathcal{D}}$, there exists a $\mathcal{B}$-vector $\hat{x}$ such that $(\hat{x};\hat{\mathcal{D}})$ is in the $\mathcal{B}$-kernel.

<u>Proof</u>: Express Theorem 2.1 in the first order predicate calculus using symbols from L. Reinterpret the statement in $\mathcal{B}$. The reinterpreted statement yields Theorem 3.1.

<u>Theorem 3.2</u>. Let $G = (N;v)$ be a finite superadditive game. Let $(x;\mathcal{D}) \in K(G)$. Then for all i in N, $x_i \leq \Omega(i)$.

<u>Proof</u>: By contradiction. Assume that there exists a player j_1 for which $x_{j_1} > \Omega(j_1)$. It is clear that $\Omega(j_1) \geq 0$. Then $x_{j_1} > 0$. Let T be the coalition in $\mathcal{D}$ for which $j_1 \in T$. T must contain more than one player. Otherwise, by (2.1), it follows that $x_{j_1} = 0$. The excess $e((T - \{j_1\}),x) > 0$ because $e((T - \{j_1\}),x) =$

$$= v(T - \{j_1\}) - \Sigma_{k\in(T-\{j_1\})}x_k = v(T) - \sigma(j_1,T) - \Sigma_{k\in(T-\{j_1\})}x_k \geq$$

$$\geq v(T) - \Omega(j_1) - \Sigma_{k\in(T-\{j_1\})}x_k > v(T) - x_{j_1} - \Sigma_{k\in T-\{j_1\}}x_k =$$

$$= v(T) - \Sigma_{k\in T}x_k = v(T) - v(T) = 0.$$

Also, the excess $e(\{j_1\},x) = v(\{j_1\}) - x_{j_1} = 0 - x_{j_1} < 0$. We assert that for any coalition S containing j_1 there exists a non-empty coalition V not containing j_1 for which $e(V) > e(S)$. We have proved this for $S = \{j_1\}$. Let S contain more than one player, then $e(S,x) < e((S - \{j_1\}),x)$ since $e(S,x) =$

$$= v(S) - \Sigma_{k\in S}x_k = v(S - \{j_1\}) + \sigma(j,S) - \Sigma_{k\in S}x_k \leq v(S - \{j_1\}) +$$

$$+ (\Omega(j_1) - x_{j_1}) - \Sigma_{k\in S-\{j_1\}}x_k < v(S - \{j_1\}) - \Sigma_{k\in S-\{j_1\}}x_k = e(S - \{j_1\},x).$$

Let V_1 be a coalition such that for each coalition V_2, $e(V_2,x) \leq e(V_1,x)$. Then $j_1 \not\in V_1$. V_1 must contain at least one player in $T - \{j_1\}$; if not, then

$$e([T - \{j_1\}] \cup V_1,x) = v([T - \{j_1\}] \cup V_1) - \Sigma_{k\in T-j_1}x_k - \Sigma_{k\in V_1}x_k$$

$$\geq v(T - \{j_1\}) - \Sigma_{k\in T-\{j_1\}}x_k + v(V_1) - \Sigma_{k\in V_1}x_k$$

$$= e(T - \{j_1\},x) + e(V_1,x) > e(V_1)$$

in contradiction to the assumption that for all V_2, $e(V_2) \leq e(V_1)$.

Let ℓ be a player contained in both V_1 and $T - \{j_1\}$. From what we have seen there exists a coalition C in $\mathcal{T}_{\ell j_1}$ (e.g., V_1) such that for any coalition D in $\mathcal{T}_{j,\ell}$, $e(c,x) > e(D,x)$. We have shown that $x_{j_1} > 0$. Then ℓ outweighs j_1. This is in contradiction to the assumption that $(x;\mathcal{D})$ is in $K(G)$. The lemma is thus proven.

Note: When $\mathscr{D} = \{N\}$ the requirement that the game be superadditive is not needed.

Lemma 3.3. Let $\hat{G} = (\hat{N};\hat{v})$ be a $\mathscr{B}$-finite superadditive game. Let $(\hat{x};\hat{\mathscr{D}})$ be a $\mathscr{B}$-i.r.p.c. in the $\mathscr{B}$-kernel of $\hat{G}$. Let $\hat{\Omega}$ be the $\mathscr{B}$-function corresponding to Ω. Then for all $\hat{i}$ in $\hat{N}$, $\hat{\Omega}(\hat{i}) \geq \hat{x}_i$.

The proof is similar to the proof of Lemma 3.1 (see Theorem 3.2).

Let $\Gamma = (N;v)$ be countably infinite, where $N = \{1,2,3,\ldots\}$ and v, the characteristic function, fulfills the following conditions:

(3.2) v is superadditive (see Definition 2.2).

(3.3) For any $0 < \varepsilon$ and for any coalition S there exists a natural number $n_1 = n_1(S,\varepsilon)$ such that for any $n \geq n_1$, $0 \leq v(S) - v(S - \{n+1, n+2, \ldots\}) < \varepsilon$.

(3.4) $\Sigma_{j=1}^{\infty}\Omega(j) < \infty$ (see (2.6)).

Let $\mathscr{D}$ be any coalition structure on Γ. Let $\hat{\Gamma} = (\hat{N};\hat{v})$ be the $\mathscr{B}$-game corresponding to Γ in $\mathscr{B}$. Let $\hat{\mathscr{D}}$ be the image of $\mathscr{D}$. Let $\hat{m}_1$ be some infinite natural $\mathscr{B}$-number.* Let

* The roofs on symbols like + > ≥ < ≤ | | (absolute value) × ∈ ∩ ∪ etc. denoting the use of the non-standard model will be omitted.

$$\hat{N}_{\hat{m}_1} = \{\hat{n} \mid \hat{n} \le \hat{m}_1,\ \hat{n} \text{ a natural } \quad \text{-number}\},$$

$$\hat{v}_{\hat{m}_1} = \text{the } \mathcal{B}\text{-function obtained from } \hat{v} \text{ by restricting its domain to be the } \mathcal{B}\text{-subsets of } \hat{N}_{\hat{m}_1},$$

$$\hat{\mathcal{D}}_{\hat{m}_1} = \{\hat{T} \mid \hat{T} = \hat{T}^1 \cap \hat{N}_{\hat{m}_1},\ \hat{T}^1 \in \hat{\mathcal{D}}\}.$$

Let $\hat{z}$ be an $\hat{m}_1$-dimensional $\mathcal{B}$-vector such that $(\hat{z};\hat{\mathcal{D}}_{\hat{m}_1})$ is a $\mathcal{B}$-i.r.p.c. of $\hat{\Gamma}_{\hat{m}_1} = (\hat{N}_{\hat{m}_1};\hat{v}_{\hat{m}_1})$ and such that

(3.5) $\quad \hat{z}_{\hat{i}} \le \hat{\Omega}(\hat{i})$, for all $\hat{i}$ such that $1^{\sim} \le \hat{i} \le \hat{m}_1$.

Let z be the infinite dimensional $\mathcal{A}$-vector defined as follows:

$$z_k = [\hat{z}_{k^\sim}]^{*v}$$

($k^\sim$ is the image of k in $\mathcal{B}$; $\hat{z}_{k^\sim}$ is the $k^\sim$-th component of $\hat{z}$; $[\hat{z}_{k^\sim}]^*$ is the nearest standard number to $\hat{z}_{k^\sim}$; $[\hat{z}_{k^\sim}]^{*v}$ is the counter image of $[\hat{z}_{k^\sim}]^*$ in $\mathcal{A}$.)

<u>Lemma 3.4</u>. For every coalition S in $\mathcal{D}$, $\Sigma_{i \in S} z_i$ converges.

<u>Proof</u>: It is clear that $z_i \ge 0$ for every natural number i. For every natural number i in $\mathcal{A}$ let $\hat{p}_{i^\sim} = [\hat{z}_{i^\sim}]^* - \hat{z}_{i^\sim}$. Let $\hat{S}$ be the image of S in $\mathcal{B}$. Then for every natural number ℓ in $\mathcal{A}$ and for all $\delta > 0$ in $\mathcal{A}$

$$\hat{0} \le [\Sigma_{\substack{i \le \ell \\ i \in S}} z_i]^{\sim} = \hat{\Sigma}_{\substack{\hat{i} \le \ell^\sim \\ \hat{i} \in \hat{S}}} [\hat{z}_{\hat{i}}]^* = \hat{\Sigma}_{\substack{\hat{i} \le \ell^\sim \\ \hat{i} \in \hat{S}}} \hat{z}_{\hat{i}} + \hat{\Sigma}_{\substack{\hat{i} \le \ell^\sim \\ \hat{i} \in \hat{S}}} \hat{p}_{\hat{i}} < \hat{\Sigma}_{\substack{\hat{i} \le \ell^\sim \\ \hat{i} \in \hat{S}}} \hat{z}_{\hat{i}} + \delta^{\sim} \cdot \hat{\Sigma}_{\substack{\hat{i} \le \ell^\sim \\ \hat{i} \in \hat{S}}} [\tfrac{1}{2}^{\sim}]^{\hat{i}}$$

This arises because for each $\hat{i}$ such that $\hat{i} \le \ell^\sim$, $\hat{i}$ is a standard number and $\hat{p}_{\hat{i}}$ is infinitesimal (positive or negative)

whereas $\delta^{\sim}\cdot[\frac{1}{2}^{\sim}]^{\hat{i}}$ is a $\hat{\text{standard}}$ positive number;[1] hence $\hat{p}_{\hat{i}} < \delta^{\sim}\cdot[\frac{1}{2}^{\sim}]^{\hat{i}}$.

$$\hat{\Sigma}_{\substack{\hat{i}\leq\ell^{\sim} \\ \hat{i}\in\hat{S}}}\hat{z}_{\hat{i}} + \delta^{\sim}\cdot\hat{\Sigma}_{\substack{\hat{i}\leq\ell^{\sim} \\ \hat{i}\in\hat{S}}}[\frac{1}{2}^{\sim}]^{\hat{i}} < \hat{\Sigma}_{\hat{i}\in\hat{S}_{\hat{m}_1}}\hat{z}_{\hat{i}} + \delta^{\sim}, \text{ where } \hat{S}_{\hat{m}_1} = \hat{S} \cap \hat{N}_{\hat{m}_1}$$

$$= \hat{v}_{\hat{m}_1}(\hat{S}_{\hat{m}_1}) + \delta^{\sim} = \hat{v}(\hat{S}_{\hat{m}_1}) + \delta^{\sim} \leq \hat{v}(\hat{S}) + \delta^{\sim}$$

Thus for every natural ℓ and all $0 < \ell$

$$\Sigma_{\substack{i\leq\ell \\ i\in S}}z_i < v(S) + \delta.$$

Theorem 3.5. Let $\hat{z}$ be an $\hat{m}_1$-dimensional $\mathcal{B}$-vector satisfying (3.5) such that $(\hat{z};\hat{\mathcal{D}}_{\hat{m}_1})$ is a $\mathcal{B}$-i.r.p.c. of $\hat{\Gamma}_{\hat{m}_1}$ and let $z_k = [\hat{z}_k\sim]^{\#v}$. Here $\hat{\Gamma}_{\hat{m}}$ is derived from (a countably infinite game Γ whose characteristic function satisfies (3.2)-(3.4). For every coalition S in $\mathcal{D}$, $\Sigma_{i\in S}z_i = v(S)$.

Proof: In the proof of Lemma 3.4 we saw that $0 \leq \Sigma_{\substack{i\in S \\ i\leq\ell}}z_i \leq v(S)$

for every natural number ℓ. What remains to be proven is that for all $\delta > 0$ there exists a natural number ℓ_1 in $\mathcal{A}$ such that $\Sigma_{\substack{i\in S \\ i\leq\ell}}z_i + \delta > v(S)$. Let ℓ_1 be a natural number such

that $\Sigma_{i>\ell}\Omega(i) \leq \frac{\delta}{3}$ and such that for all $n > \ell_1$,
$v(S) - v(S - \{n+1, n+2, \ldots\}) \leq \frac{\delta}{3}$ (see (3.3)-(3.4)). Then

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(1) The meaning of $\hat{=}$ in the non-standard model is exactly $=$; hence we are justified in writing $=$ instead of $\hat{=}$. This is because $\emptyset_{ID}$ is the identity relation.

$$\Big(\Sigma_{\substack{i\in S\\ i\le \ell_1}} z_i + \tfrac{2}{3}\delta\Big)^\sim > \Big(\Sigma_{\substack{i\in S\\ i\le \ell_1}} z_i + \tfrac{\delta}{3}\cdot \Sigma_{\substack{i\in S\\ i\le \ell_1}} (\tfrac{1}{2})^i\Big)^\sim + \Big(\Sigma_{i>\ell_1}\Omega(i)\Big)^\sim$$

$$= \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \hat z_{\hat i}^{*} + \big(\tfrac{\delta}{3}\big)^\sim \cdot \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \big(\tfrac{1}{2}^\sim\big)^{\hat i} + \hat\Sigma_{\hat i>\ell_1^\sim}\hat\Omega(\hat i)$$

$$= \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} (\hat z_{\hat i} + \hat p_{\hat i}) + \big(\tfrac{\delta}{3}\big)^\sim \cdot \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \big(\tfrac{1}{2}^\sim\big)^{\hat i} + \hat\Sigma_{\hat i>\ell_1^\sim}\hat\Omega(\hat i)$$

where $\hat p_{\hat i} = \hat z_{\hat i}^{*} - \hat z_{\hat i}$, and is, of course, infinitesimal (positive or negative). Thus $(\frac{\delta}{3})^\sim \cdot (\frac{1}{2}^\sim) + \hat p_{\hat i} > \hat 0$ for all $\hat i \le \ell_1^\sim$. Hence

$$\hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} (\hat z_{\hat i} + \hat p_{\hat i}) + \big(\tfrac{\delta}{3}\big)^\sim \cdot \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \big(\tfrac{1}{2}^\sim\big)^{\hat i} + \hat\Sigma_{\hat i>\ell_1^\sim}\hat\Omega(\hat i) \ge \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \hat z_{\hat i} + \hat\Sigma_{i>\ell_1^\sim}\hat\Omega(\hat i)$$

and by (3.5)

$$\ge \hat\Sigma_{\substack{\hat i\in\hat S\\ \hat i\le \ell_1^\sim}} \hat z_{\hat i} + \hat\Sigma_{\substack{\hat i>\ell_1^\sim\\ \hat i\in\hat S_{\hat m_1}}} \hat z_{\hat i}$$

and since $(\hat z;\hat{\mathcal{D}}_{\hat m_1})$ is a $\mathcal{B}$-i.r.p.c. and $\hat S_{\hat m_1}$ is in $\hat{\mathcal{D}}_{\hat m_1}$ we have

$$= \hat v_{\hat m_1}(\hat S_{\hat m_1}) = \hat v(\hat S_{\hat m_1}).$$

For all $\hat n > \ell_1^\sim$ we know that $\hat v(\hat S) - \hat v(\hat S - \{\hat n + 1^\sim, \hat n + 2^\sim, \ldots\}) \le (\frac{\delta}{3})^\sim$. Then $\hat v(\hat S) - \hat v(\hat S_{\hat m_1}) \le (\frac{\delta}{3})^\sim$. Then $\hat v(\hat S_{\hat m_1}) \ge \hat v(\hat S) - (\frac{\delta}{3})^\sim$.

From this we deduce that

$$(\Sigma_{\substack{i \in S \\ i \leq \ell_1}} z_i + \delta)^{\sim} > \hat{v}(\hat{S}).$$

Therefore,

$$\Sigma_{\substack{i \in S \\ i \leq \ell_1}} z_i + \delta > v(S).$$

<u>Lemma 3.6</u>. Let $\hat{S}^1$ be a $\mathcal{B}$-subset of $\hat{N}_{\hat{m}_1}$. Let S be the $\mathcal{A}$-coalition containing every natural $\mathcal{A}$-number j for which $j^{\sim}$ is in $\hat{S}^1$. Let $S^{\sim}$ be the image of S in $\mathcal{B}$. Then $|\hat{v}(\hat{S}^1) - v(S^{\sim})|$ is infinitesimal.

<u>Proof:</u> Let $\hat{\varepsilon}$ be a standard number greater than $0^{\sim}$. Let $\hat{\varepsilon}^{v}$ be the image of $\hat{\varepsilon}$ in $\mathcal{A}$. Let n_1 be a natural $\mathcal{A}$-number such that for all $n \geq n_1$, $|v(S_n) - v(S)| < (\frac{\hat{\varepsilon}}{3})^{v}$ and $\sum_{i \geq n_1} \Omega(i) < (\frac{\hat{\varepsilon}}{3})^{\wedge}$; here $S_n = S - \{n+1, n+2, \ldots\}$ (see (3.3)). Then for any standard $\mathcal{B}$-natural number $\hat{n}$ greater than $\hat{n}_1$, $|\hat{v}(S^{\sim}_{\hat{n}}) - \hat{v}(S^{\sim})| < (\frac{\hat{\varepsilon}}{3})$. Since $S^{\sim}_{\hat{n}}$ and $\hat{S}^1_{\hat{n}}$ coincide for all standard $\hat{n}$, this means that for all standard $\hat{n}$ greater than $n_1^{\sim}$,

(3.6) $$|\hat{v}(\hat{S}^1_{\hat{n}}) - \hat{v}(S^{\sim})| < (\frac{\hat{\varepsilon}}{3})$$

Suppose $|\hat{v}(\hat{S}^1) - \hat{v}(S^{\sim})| > \hat{\varepsilon}$. If $\hat{v}(S^{\sim}) > \hat{v}(\hat{S}^1)$ then $v(S^{\sim}) - \hat{v}(\hat{S}^1) > \hat{\varepsilon}$. Because of superadditivity

$$\hat{v}(\hat{S}^1) \geq v(\hat{S}^1_{\hat{n}}) + \hat{v}(\hat{S}^1 - \hat{S}^1_{\hat{n}}).$$

Hence

$$v(S^{\sim}) - \hat{v}(\hat{S}^1_{\hat{n}}) \geq \hat{v}(S^{\sim}) - (\hat{v}(\hat{S}^1_{\hat{n}}) + \hat{v}(\hat{S}^1 - \hat{S}^1_{\hat{n}})) \geq \hat{v}(S^{\sim}) - \hat{v}(\hat{S}^1) > \hat{\varepsilon}$$

which contradicts (3.6). Thus if $|\hat{v}(\hat{S}^1) - \hat{v}(\hat{S}^{\sim})| > \hat{\varepsilon}$ then

$$\hat{v}(\hat{S}^1) > \hat{v}(S^{\sim}) \tag{3.7}$$

Let $W = W_1 \cup W_2$ be any finite $\mathcal{A}$-coalition, where W_1 and W_2 are disjoint subsets of W. Through mathematical induction, and using (2.6), it may be readily seen that $v(W) \le v(W_1) + \Sigma_{i\in W_2}\Omega(i)$. The assertion that $v(W) \le v(W_1) + \Sigma_{i\in W_2}\Omega(i)$ for all W, W_1, W_2 such that $W = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$, and $\exists n\, \forall i\, (i \in W \to i < n)$, is expressible as a sentence in the first order predicate calculus. Since this sentence is true in $\mathcal{A}$ it is also true when re-interpreted in $\mathcal{B}$. Applying this knowledge to $\hat{S}^1$ we obtain:

$$\hat{v}(\hat{S}^1{}_{n_1^{\sim}}) \le \hat{v}(\hat{S}^1) \le \hat{v}(\hat{S}^1{}_{n_1^{\sim}}) + \hat{\Sigma}_{\hat{i}\in\hat{S}^1-\hat{S}^1_{n^{\sim}}}\hat{\Omega}(\hat{i}) \le \hat{v}(\hat{S}^1_{n_1^{\sim}}) + \tag{3.8}$$

$$+ \hat{\Sigma}_{\hat{i}\ge n_1^{\sim}}\hat{\Omega}(\hat{i}) \le \hat{v}(\hat{S}^1_{n_1^{\sim}}) + (\tfrac{\hat{\varepsilon}}{3}).$$

By superadditivity,

$$\hat{v}(S^{\sim}) \ge \hat{v}(\hat{S}^1{}_{n_1^{\sim}}) \tag{3.9}$$

Using (3.7), (3.8) and (3.9) we receive:

$|\hat{v}(\hat{S}^1) - \hat{v}(\hat{S}^{\sim})| = \hat{v}(\hat{S}^1) - \hat{v}(S^{\sim}) \le (\hat{v}(\hat{S}^1{}_{n_1^{\sim}}) + (\tfrac{\hat{\varepsilon}}{3})) - \hat{v}(\hat{S}^1{}_{n_1^{\sim}}) = (\tfrac{\hat{\varepsilon}}{3})$.

This is in contradiction to the supposition that $|\hat{v}(\hat{S}^1) - \hat{v}(S^{\sim})| > \hat{\varepsilon}$. Thus for every standard $\mathcal{B}$-number, $\hat{\varepsilon}$, $\hat{\varepsilon} > 0^{\sim}$, $|\hat{v}(\hat{S}^1) - \hat{v}(S^{\sim})| \le \hat{\varepsilon}$. Hence $|\hat{v}(\hat{S}^1) - \hat{v}(S^{\sim})|$ is infinitesimal.

<u>Lemma 3.7</u>. Let $\hat{S}^1$ be a $\mathcal{B}$-subset of $\hat{N}_{\hat{m}_1}$. Let S be the $\mathcal{A}$-coalition containing every natural $\mathcal{A}$-number j for which $j^{\sim}$ is in $\hat{S}^1$. Then $|\hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} - (\Sigma_{i\in S}z_i)^{\sim}|$ is infinitesimal. Here $\hat{z}$ and z are as defined in the paragraph containing (3.5).

Proof: Since $\Sigma_{i=1}^{\infty} z_i \leq v(\{1,2,\ldots\}) < \infty$, it is clear then that for any $\mathcal{A}$-coalition T $\Sigma_{i\in T} z_i$ converges absolutely. Let δ be any particular positive $\mathcal{A}$-number and let ℓ be a natural $\mathcal{A}$-number such that $\Sigma_{i\geq \ell}\Omega(i) < \frac{\delta}{3}$ and such that $\Sigma_{\substack{i\in S\\ i\geq \ell}} z_i < \frac{\delta}{3}$. Then

$$\begin{aligned}
|\hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} - (\Sigma_{i\in S} z_i)^{\sim}| &= |(\hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}\leq\ell^{\sim}}}\hat{z}_{\hat{i}} - \hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}\leq\ell^{\sim}}}\hat{z}_{\hat{i}}^{*}) + (\hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}>\ell^{\sim}}}\hat{z}_{\hat{i}} - (\Sigma_{\substack{i\in S\\ i>\ell}} z_i)^{\sim})| \\
&\leq |\hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}\leq\ell^{\sim}}}\hat{z}_{\hat{i}} - \hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}\leq\ell^{\sim}}}\hat{z}_{\hat{i}}^{*}| + \hat{\Sigma}_{\substack{\hat{i}\in\hat{S}^1\\ \hat{i}>\ell^{\sim}}}\hat{z}_{\hat{i}} + (\Sigma_{\substack{i\in S\\ i>\ell}} z_i)^{\sim} \\
&\leq |\hat{\epsilon}_1 + \ldots + \hat{\epsilon}_{\ell^{\sim}}| + (\tfrac{\delta}{3})^{\sim} + (\tfrac{\delta}{3})^{\sim} \\
&\qquad \text{where } \hat{\epsilon}_1, \ldots, \hat{\epsilon}_{\ell} \text{ are infinitesimal numbers,} \\
&< (\tfrac{\delta}{3})^{\sim} + (\tfrac{\delta}{3})^{\sim} + (\tfrac{\delta}{3})^{\sim} = \delta^{\sim}
\end{aligned}$$

Thus for every standard positive $\mathcal{B}$-number $\delta^{\sim}$, $|\hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} - (\Sigma_{i\in S} z_i)^{\sim}|$ is less than it. Therefore $|\hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} - (\Sigma_{i\in S} z_i)^{\sim}|$ is infinitesimal.

Theorem 3.8 (an existence theorem): Let $\Gamma = (N;v)$ be an infinite game where v, the characteristic function, fulfills conditions (3.2), (3.3) and (3.4). Then, for any coalition structure $\mathcal{D}$ there exists an infinite dimensional vector x such that $(x;\mathcal{D})$ is in $K(\Gamma)$.

Proof: Let $\hat{\Gamma} = (\hat{N};\hat{v})$ be the $\mathcal{B}$-game corresponding to Γ in $\mathcal{B}$. Let $\hat{\mathcal{D}}$ be the image of $\mathcal{D}$. Let $\hat{m}_1$, $\hat{N}_{\hat{m}_1}$, $\hat{v}_{\hat{m}_1}$, and $\hat{\mathcal{D}}_{\hat{m}_1}$ be as defined in the lines following (3.4). Let z be an $\hat{m}_1$ dimensional $\mathcal{B}$-vector such that $(z;\hat{\mathcal{D}}_{\hat{m}_1})$ is in the $\mathcal{B}$-kernel of $\hat{\Gamma}_{\hat{m}_1}$.

Such a $\hat{z}$ exists, by Lemma 3.1. For all finite natural $\hat{i}$, $\hat{z}_{\hat{i}}$ is finite since by Lemma 3.3 $\hat{z}_{\hat{i}} \leq \hat{\Omega}(\hat{i})$, and $\hat{\Omega}(\hat{i})$ is finite. We define the infinite dimensional $\mathcal{A}$-vector z as follows: $z_i = [\hat{z}_i\sim]^{*v}$. It is clear than $z_i \geq 0$ for all natural i. By Therem 3.5 we know that $\Sigma_{i\in S} z_i = v(S)$ for all S in $\mathcal{D}$. Thus $(z;\mathcal{D})$ is an i.r.p.c. We seek to prove that $(z;\mathcal{D})$ is in the kernel of Γ.

Let H be a coalition in $\mathcal{D}$ that contains at least two different players. It is sufficient to prove that at least one of the following two cases holds:

(i) $z_\ell = 0$

(ii) For any coalition K that contains k and does not contain ℓ, there exists a coalition S which contains ℓ and which does not contain k, such that

$$v(S) - \Sigma_{i\in S} z_i \geq v(K) - \Sigma_{i\in K} z_i$$

We shall prove that when (i) does not hold, (ii) does. Let K be some coalition that contains player k and does not contain player ℓ. Let $K^\sim$ be the image of K in $\mathcal{B}$. Let $K^{\sim\hat{}}_{m_1} = K^\sim \cap \hat{N}^{\hat{}}_{m_1}$. $K^{\sim\hat{}}_{m_1}$ contains the player $k^\sim$ and does not contain the player $\ell^\sim$. Let $H^\sim$ be the image of H and let $H^{\sim\hat{}}_{m_1} = H^\sim \cap \hat{N}^{\hat{}}_{m_1}$. Since $(\hat{z};\hat{\mathcal{D}}^{\hat{}}_{m_1})$ is in the $\mathcal{B}$-kernel and since $k^\sim, \ell^\sim \in H^{\sim\hat{}}_{m_1} \in \mathcal{D}^{\hat{}}_{m_1}$ and $\hat{z}_\ell\sim > 0^\sim$, there exists a $\mathcal{B}$-coalition $\hat{S}^1$ of $\hat{\Gamma}^{\hat{}}_{m_1}$ that contains $\ell^\sim$ and does not contain $k^\sim$ and for which

$$\hat{v}_{\hat{m}_1}(\hat{S}^1) - \hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} \geq \hat{v}_{\hat{m}_1}(\mathcal{K}^{\sim}_{\hat{m}_1}) - \hat{\Sigma}_{\hat{i}\in\mathcal{K}^{\sim}_{\hat{m}_1}}\hat{z}_{\hat{i}}$$

Let S be the $\mathcal{A}$-coalition containing every natural $\mathcal{A}$-number j for which $j^{\sim}$ is in $\hat{S}^1$. It is clear that S contains ℓ and does not contain k. Let $S^{\sim}$ be the image of S in $\mathcal{B}$. Note that $S^{\sim}$ and $\hat{S}^1$ are in general not identical. $\hat{S}^1$ contains only $\mathcal{B}$-numbers that are less than $\hat{m}_1 + 1^{\sim}$. $S^{\sim}$, on the other hand, may contain greater $\mathcal{B}$-numbers. We set out to prove that $v(S) - \Sigma_{i\in S}z_i \geq v(\mathcal{K}) - \Sigma_{i\in\mathcal{K}}z_i$. By Lemma 3.6, $|\hat{v}(\hat{S}^1) - \hat{v}(S^{\sim})|$ and $|\hat{v}(\mathcal{K}^{\sim}_{\hat{m}_1}) - \hat{v}(\mathcal{K}^{\sim})|$ are infinitesimal numbers. (The latter difference is infinitesimal because both coalitions have the same standard players.) By Lemma 3.7, $|\hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} - (\Sigma_{i\in S}z_i)^{\sim}|$ and $|\hat{\Sigma}_{\hat{i}\in\mathcal{K}^{\sim}_{\hat{m}_1}}\hat{z}_{\hat{i}} - (\Sigma_{i\in\mathcal{K}}z_i)^{\sim}|$ are likewise infinitesimal. To prove that $(v(\mathcal{K}) - \Sigma_{i\in\mathcal{K}}z_i) \leq (v(S) - \Sigma_{i\in S}z_i)$ it is sufficient to prove that for all $\delta > 0$ in $\mathcal{A}$,

$$(v(\mathcal{K}) - \Sigma_{i\in\mathcal{K}}z_i) - (v(S) - \Sigma_{i\in S}z_i) \leq \delta.$$

But $(v(\mathcal{K}) - \Sigma_{i\in\mathcal{K}}z_i)^{\sim} - (v(S) - \Sigma_{i\in S}z_i)^{\sim} < \hat{v}(\mathcal{K}^{\sim}_{\hat{m}_1}) + (\frac{\delta}{4})^{\sim} -$

$$- \hat{\Sigma}_{\hat{i}\in\mathcal{K}^{\sim}_{\hat{m}_1}}\hat{z}_{\hat{i}} + (\tfrac{\delta}{4})^{\sim} - \hat{v}(\hat{S}^1) + (\tfrac{\delta}{4})^{\sim} + \hat{\Sigma}_{\hat{i}\in\hat{S}^1}\hat{z}_{\hat{i}} + (\tfrac{\delta}{4})^{\sim} \leq \delta^{\sim}.$$

We have thus proven that $(z;\mathcal{D})$ is in the kernel. Therefore the kernel is not empty for any coalition structure.

<u>Theorem 3.9</u>. Let $K(G)$ be the kernel of an infinite game $G = (\{1,2,\ldots\};v)$ which fulfills the relations (3.2), (3.3) and (3.4). Let $\mathcal{D}$ be an arbitrary coalition structure on G. Let G_n be the game $(\{1,2,\ldots,n\};v_n)$, where v_n receives the same values as v on subsets of $\{1,2,\ldots,n\}$. Let $K(G_n)$ be

the kernel of G_n. Let the space $E^\infty = E^1 \times E^1 \times \ldots$ have the Tychinoff topology. Let $\{O_i\}_{i=1,2,\ldots}$ be a sequence of sets, $O_i \subset E^i$. Let O_∞ be a set in the space E^∞ with the following property: If $x = (x_1, x_2, \ldots)$ is a point in E^∞ such that for any open set E containing x there exists a natural number i and a vector $(x_1', x_2', \ldots, x_i')$ in O_i such that $(x_1', x_2', \ldots, x_i', 0, 0, \ldots) \in E$ then $x \in O_\infty$. Under these conditions, if for each n, there exists a vector $x^{(n)} = (x_1^{(n)}, \ldots, x_n^{(n)})$ in O_n such that $(x^{(n)}; \mathcal{D}_n)$ is in $K(G_n)$, then there exists an x in O_∞ such that $(x; \mathcal{D}) \in K(G)$.

Proof: Since for each n there exists an $x^{(n)}$ in O_n for which $(x^{(n)}; \mathcal{D}_n)$ is in $K(G_n)$, and since this fact is expressible in the first order predicate calculus, it follows that for any natural $\mathcal{B}$-number, $\hat{m}$, there exists an $\hat{x}^{(\hat{m})}$ in $\hat{O}_{\hat{m}}$ such that $(\hat{x}^{(\hat{m})}; \hat{\mathcal{D}}_{\hat{m}})$ is in $\hat{K}(\hat{G}_{\hat{m}})$. Let $\hat{m}_1$ be an infinite $\mathcal{B}$-number. Let $\hat{x}^{(\hat{m}_1)}$ be such that $\hat{x}^{(\hat{m}_1)} \in \hat{O}_{\hat{m}_1}$ and $(\hat{x}^{(\hat{m}_1)}; \hat{\mathcal{D}}_{\hat{m}_1}) \in \hat{K}(\hat{G}_{\hat{m}_1})$. Let x be the infinite dimensional $\mathcal{A}$-vector obtained by setting $x_i = (\hat{x}^{(\hat{m}_1)}{}_{i^\sim}{}^{\bullet})^{v}$. Then, as we have shown in the proof of Theorem 3.8, $(x; \mathcal{D}) \in K(G)$. We must prove that $x \in O_\infty$. Let $x^\sim$ be the image in $\mathcal{B}$ of x. We will prove that for all ϵ, $\hat{\Sigma}_{i \le \hat{m}_1} |x^\sim{}_i - \hat{x}_i| + \hat{\Sigma}_{i > \hat{m}_1} x^\sim{}_i < \epsilon^\sim$. Let j_1 be a natural $\mathcal{A}$-number for which $\Sigma_{i > j_1} \Omega(i) < \frac{1}{4}\epsilon$ and for which $\Sigma_{i > j_1} x_i < \frac{1}{4}\epsilon$. Then by Lemma 3.3 $\hat{\Sigma}_{\hat{m}_1 \ge i > j_1^\sim} \hat{x}^{(\hat{m}_1)}{}_i < (\frac{1}{4})^\sim \cdot \epsilon^\sim$ and $\hat{\Sigma}_{i > j_1^\sim} x^\sim_i < (\frac{1}{4})^\sim \cdot \epsilon^\sim$. Thus

$$(3.10)\quad \hat{\Sigma}_{i \le \hat{m}_1} |x^\sim{}_i - \hat{x}^{(\hat{m}_1)}{}_i| + \hat{\Sigma}_{i > \hat{m}_1} x^\sim{}_i \le \hat{\Sigma}_{i \le j_1^\sim} |x^\sim{}_i - \hat{x}^{(\hat{m}_1)}{}_i| + \hat{\Sigma}_{i > j_1^\sim} x^\sim{}_i + \hat{\Sigma}_{i > j^\sim} \hat{x}^{(\hat{m}_1)}{}_i < \hat{\delta} + (\tfrac{1}{4})^\sim \cdot \epsilon^\sim + (\tfrac{1}{4})^\sim \cdot \epsilon^\sim$$

$\hat{\delta}$ is infinitesimal. Hence

$$\hat{\delta} + (\tfrac{1}{4})^{\sim} \cdot \epsilon^{\sim} + (\tfrac{1}{4})^{\sim} \cdot \epsilon^{\sim} < \epsilon^{\sim}.$$

Thus if $\hat{\epsilon}$ is a standard number, then there exists an $\hat{i}_1$ (in our case $\hat{i}_1 = \hat{m}_1$), and a vector $\hat{x}^{(\hat{i}_1)}$ in $\hat{O}_{\hat{i}_1}$ such that $\hat{\Sigma}_{i \le \hat{i}_1} |x^{\sim}_i - \hat{x}^{(\hat{i}_1)}_i| + \hat{\Sigma}_{i > \hat{i}_1} x^{\sim}_i < \hat{\epsilon}$. For any specific standard $\hat{\epsilon}$ ($\hat{\epsilon} = \tfrac{1}{2}^{\sim}, \tfrac{1}{3}^{\sim}, \ldots,$ etc.) the phrase: "There exists an $\hat{i}_1$ and a vector $\hat{x}^{(\hat{i}_1)}$ in $\hat{O}_{\hat{i}_1}$ such that

$$\hat{\Sigma}_{i \le \hat{i}_1} |x^{\sim}_i - \hat{x}^{(\hat{i}_1)}_i| + \hat{\Sigma}_{i > \hat{i}_1} x^{\sim}_i < \hat{\epsilon}."$$

is expressible as a sentence in the first order predicate calculus. This sentence is true in $\mathcal{B}$ for each specific standard $\hat{\epsilon} > 0^{\sim}$. Then the sentence must be true in $\mathcal{A}$ for any specific $\epsilon > 0$. Thus for any $\epsilon > 0$ there exists an i_1 and an $x^{(i_1)}$ in O_{i_1} such that

$$\Sigma_{i \le i_1} |x_i - x^{(i_1)}_i| + \Sigma_{i > i_1} x_i < \epsilon. \tag{3.11}$$

This means that for any open set E containing x there exists an i_1 and a vector $(x^{(i_1)}_1, \ldots, x^{(i_1)}_{i_1})$ such that $(x^{(i_1)}_1, \ldots, x^{(i_1)}_{i_1}, 0, 0, \ldots)$ is in E and $(x^{(i_1)}_1, \ldots, x^{(i_1)}_{i_1}) \in O_{i_1}$. Then by the conditions of the theorem, $x \in O_\omega$.

Clearly, Theorem 3.8 is a special case of Theorem 3.9.

Theorem 3.9 is useful for extending known theorems about the kernel of finite games to infinite games. For example, it is known (see [2]) that if a finite game has a non-empty core then the kernel intersects the core. (The same is true if "core" is replaced by "pseudo-core" (see [2]).) It follows from Theorem 3.9 that the same result holds for games with a countable number of

players, if the characteristic function, v, satisfies (3.2), (3.3), and (3.4).

<u>Alternative Proof of Theorem 3.9</u>. (Suggested by R. J. Aumann.) For each ℓ, $\ell = 1,2,\ldots$, let $x^{(\ell)}$ be an ℓ-dimensional vector such that $(x^{(\ell)};\mathcal{D}_\ell) \in K(G_\ell)$ and $x^{(\ell)} \in O_\ell$. For all $\ell \geq 1$ and for all k, $1 \leq k \leq \ell$, $x_k^{(\ell)} \leq v_\ell(\{1,\ldots,\ell\}) = v(\{1,\ldots,\ell\}) \leq v(N)$. Denote $c = v(N)$ and let $I = [0,c] \times [0,c] \times \ldots$. Let $x'^{(\ell)}$ be the infinite dimensional vector with $x'^{(\ell)}_k = x^{(\ell)}_k$ for the first ℓ components and $x'^{(\ell)}_k = 0$ for the remaining components. Under the Tychinoff topology, I is a compact space. Then there exists a vector x in I which is a limit point of the $x'^{(\ell)}$'s. Since, by Theorem 3.2, for all k and all ℓ, $\ell \geq k$, $x_k^{(\ell)} \leq \Omega(k)$, it follows that

$$x_k \leq \Omega(k) \quad \text{for all} \quad k \geq 1 \tag{3.12}$$

Let $C \in \mathcal{D}$ and let $\epsilon > 0$. We wish to show that

$$|v(C) - \Sigma_{k\in C} x_k| \leq \epsilon .$$

Let n_1 be such that

$$\Sigma_{k\geq n_1} \Omega(k) \leq \tfrac{1}{4}\epsilon \tag{3.13}$$

and such that for all $n \geq n_1$,

$$v(C) - v(C_n) \leq \tfrac{1}{4}\epsilon . \tag{3.14}$$

Condition (3.4) assures the existence of such an n_1. Let m_1 be greater than n_1 and be large enough so that

$$\Sigma_{1\leq k\leq n_1} |x_k^{(m_1)} - x_k| \leq \tfrac{1}{4}\epsilon .$$

By Theorem 3.2, $x_k^{(m_1)} \le \Omega(k)$ for all k, $1 \le k \le m_1$. Thus, by (3.12) and (3.13),

$$(3.15) \qquad \Sigma_{1 \le k \le m_1} |x_k^{(m_1)} - x_k| + \Sigma_{k>m_1} x_k \le \tfrac{1}{2}\epsilon.$$

Since $v(C_{m_1}) - \Sigma_{k \in C_{m_1}} x_k^{(m_1)} = 0$, we may readily derive, using (3.12), (3.13), (3.14), and (3.15), that $|v(C) - \Sigma_{k \in C} x_k| < \epsilon$. Due to the fact that ϵ is an arbitrary positive quantity, it follows that $v(C) = \Sigma_{k \in C} x_k$. $(x;\mathcal{D})$ is therefore an i.r.p.c.

Let $i,j \in C$ be two different players in C. Suppose $x_j = 0$. Let C_i be a coalition containing i and not j. To prove that $(x;\mathcal{D}) \in K(G)$ we must show the existence of a C_j, $C_j \in \mathcal{T}_{ji}$ (see (2.3)) such that $e(C_j;x) \ge e(C_i;x)$. Denote by $C_{i;n}$ the coalition C_i restricted to the first n players, for $n \ge i,j$. Let $\{x^{(n_\nu)}\}_{\nu=1,2,\ldots}$ be a sub-sequence of n_ν-dimensional vectors such that for all ν, $\nu = 1,2,\ldots,$ $(x^{(n_\nu)};\mathcal{D}_{n_\nu}) \in K(G_{n_\nu})$ and $x^{(n_\nu)} \in O_{n_\nu}$, and such that $\lim_{\nu\to\infty} x'^{(n_\nu)} = x$, where $x'^{(n_\nu)}_k = x^{(n_\nu)}_k$ if $k \le n_\nu$ and $x'^{(n_\nu)}_k = 0$ otherwise. Since $x'^{(n_\nu)} \to x$, and since $x_j > 0$, there exists a number ν_1 such that $n_{\nu_1} \ge i,j$, and such that for all $\nu \ge \nu_1$, $x_j^{(n_\nu)} > 0$. For each ν equal or greater than ν_1 there exists a coalition $C_j^{(n_\nu)}$, $C_j^{(n_\nu)} \subset \{1,2,\ldots,n_\nu\}$, such that $e(C_j^{(n_\nu)};x^{(n_\nu)}) \ge e(C_{i;n_\nu};x^{(n_\nu)})$. This is because $(x_{n_\nu};\mathcal{D}_{n_\nu}) \in K(G_{n_\nu})$.

For any coalition E, let χ_E be the $0-1$ characteristic function of the set E, i.e., $\chi_E(n) = 1$ if $n \in E$; $\chi_E(n) = 0$ otherwise. We shall now define a function on any two $0-1$ characteristic functions.

$$\rho(\chi_E,\chi_F) \overset{\text{def}}{=\!=\!=} \Sigma_{n=1}^{\infty} |\chi_E(n) - \chi_F(n)| \left(\tfrac{1}{2}\right)^n$$

One may easily verify that ρ is a metric; hence ρ induces a topology on the "space" of $0 - 1$ characteristic functions. It is easily seen that the space X of all the $0 - 1$ characteristic functions (regarded as infinite sequences) with the topology based on this metric is a compact subspace of $J = [0,1] \times [0,1] \times \ldots$, where the topology of J is the Tychinoff topology. Let $\{C_j^{(m_\nu)}\}$ be a sub-sequence of the $C_j^{(n_\nu)}$'s such that $\{\chi_{C_j^{(m_\nu)}}\}$ converges under the ρ-topology to a single limiting $0 - 1$ characteristic function. Denote the coalition corresponding to the limiting $0 - 1$ characteristic function by C_j. It is clear that C_j contains j and does not contain i. We wish to prove that $e(C_j;x) \geq e(C_i;x)$. Let ϵ be an arbitrary positive number. Let ν_1 be such that

(3.16) $$n_{\nu_1} \geq i,j$$

(3.17) $$\text{For all } \nu \geq \nu_1,\ x^{(n_{\nu_1})} > 0$$

(3.18) $$\Sigma_{k \geq n_{\nu_1}} \Omega(k) \leq \frac{1}{16}\epsilon$$

(3.19) $$\text{For all } \nu \geq \nu_1,\ \Sigma_{1 \leq k \leq n_{\nu_1}} |x_k - x_k^{(n_\nu)}| \leq \frac{1}{16}\epsilon$$

(3.20) $$\text{For all } n \geq n_{\nu_1},\ v(C_j) - v(C_{j;n}) \leq \frac{1}{16}\epsilon$$

$$\text{where } C_{j;n} = C_j \cap \{1,2,\ldots,n\}$$

(3.21) $$\text{For all } n \geq n_{\nu_1},\ v(C_i) - v(C_{i;n}) \leq \frac{1}{16}\epsilon$$

Let $C_j^{(m_o)}$ be a member of $\{C_j^{(m_\nu)}\}$ such that $C_j^{(m_o)} \cap C_{j;n_{\nu_1}} = C_{j;n_{\nu_1}}$ and such that $m_o \geq n_{\nu_1}$. It is clear

that such a $C_j^{(m_o)}$ exists because any $C_j^{(m)}$ for which $\chi_{C_j^{(m)}}$ is sufficiently close to χ_{C_j} (under the metric ρ) is bound to contain all players contained in $C_{j;n_{\nu_1}}$. Since $C_j^{(m_o)} \cap C_{j;n_{\nu_1}} = C_{j;n_{\nu_1}}$ it is easy to deduce, using (3.18), that

$$(3.22) \qquad v(C_j^{(m_o)}) - v(C_{j;n_{\nu_1}}) \le \frac{1}{16}\epsilon$$

We know that $e(C_j^{(m_o)};x^{(m_o)}) - e(C_{i;m_o};x^{(m_o)}) \ge 0$. Hence,

$$(3.23) \quad v(C_j^{(m_o)}) - \Sigma_{k\in C_j^{(m_o)}} x_k^{(m_o)} - (V(C_{i;m_o}) - \Sigma_{k\in C_{i;m_o}} x_k^{(m_o)}) \ge 0.$$

By applying standard procedure to inequality (3.23) one easily derives, by using inequalities (3.16) - (3.22), that

$$v(C_j) - \Sigma_{k\in C_j} x_k - (v(C_i) - \Sigma_{k\in C_i} x_k) \ge -\epsilon.$$

Since ϵ is an arbitrary positive quantity, this means that $v(C_j) - \Sigma_{k\in C_j} x_k \ge v(C_i) - \Sigma_{k\in C_i} x_k$, or $e(C_j;x) \ge e(C_i;x)$.

Thus $(x;\mathcal{D}) \in K(G)$. Since x is a limit point of a series of vectors $\{x^{(n)}\}$, such that for all n, $x^{(n)} \in O_n$, then by the conditions of the theorem it follows that $x \in O_\infty$.

The following theorem is an example which shows how non-standard models may generate theorems concerning the kernel of infinite games.

<u>Theorem 3.10</u>. Let $G = (N;v)$ be an infinite game satisfying (3.2), (3.3) and (3.4). Let $\mathcal{D}$ be an arbitrary coalition structure. Then for any $\epsilon > 0$ there exists an n_1 such that for any n_2 greater than n_1 and any $x^{(n_2)}$ for which $(x^{(n_2)};\mathcal{D}_{n_2}) \in K(G_{n_2})$

there exists an $n_3 < n_1$ and an $x^{(n_3)}$ such that $(x^{(n_3)};\mathcal{D}_{n_3}) \in K(G_{n_3})$ and $\Sigma_{1\leq i\leq n_3} |x_i^{(n_3)} - x_i^{(n_2)}| + \Sigma_{n_3<i\leq n_2} x_i^{(n_2)} \leq \epsilon$.

Proof: Let ϵ be an arbitrary positive quantity. Let $\hat{G}$ be the image of G in $\mathcal{B}$. Let $\hat{n}_1$ be an infinite natural number. Let $\hat{n}_2$ be an arbitrary infinite natural number such that $\hat{n}_2 > \hat{n}_1$. Let $\hat{x}^{(\hat{n}_2)}$ be such that $(\hat{x}^{(\hat{n}_2)};\hat{\mathcal{D}}_{\hat{n}_2}) \in \hat{K}(\hat{G}_{\hat{n}_2})$. Let x be an infinite dimensional $\mathcal{A}$-vector such that $x_i = (\hat{x}^{(\hat{n}_2)}{}_{i^\sim})^{*\vee}$. Let $x^\sim$ be the image of x. We have seen in the proof of Theorem 3.9 (see (3.10)) that for any positive $\mathcal{A}$-number δ,

$$(3.24) \qquad \hat{\Sigma}_{1^\sim\leq\hat{i}\leq\hat{n}_2} |x^\sim_{\hat{i}} - \hat{x}^{(\hat{n}_2)}{}_{\hat{i}}| + \hat{\Sigma}_{\hat{i}>\hat{n}_2} x^\sim_{\hat{i}} < \delta^\sim.$$

We have also seen in the same proof (see (3.11)) that there exists a natural $\mathcal{A}$-number, n_3, and a vector $x^{(n_3)}$ such that $(x^{(n_3)};\mathcal{D}_{n_3}) \in K(G_{n_3})$ and $\Sigma_{1\leq i\leq n_3} |x_i - x_i^{(n_3)}| + \Sigma_{i>n_3} x_i < \delta$. Thus

$$(3.25) \qquad \hat{\Sigma}_{1^\sim\leq\hat{i}\leq n_3^\sim} |x^\sim_{\hat{i}} - (x^{(n_3)})^\sim_{\hat{i}}| + \Sigma_{\hat{i}>n_3^\sim} x^\sim_{\hat{i}} < \delta^\sim$$

Combining (3.24) and (3.25) and setting $\delta = \frac{1}{2}\epsilon$, we receive

$$(3.26) \quad \hat{\Sigma}_{1^\sim\leq\hat{i}\leq n_3^\sim} |(x^{(n_3)})^\sim_{\hat{i}} - \hat{x}^{(\hat{n}_2)}{}_{\hat{i}}| + \hat{\Sigma}_{n_3^\sim<\hat{i}\leq\hat{n}_2} \hat{x}^{(\hat{n}_2)}{}_{\hat{i}} < 2^\sim \cdot \delta^\sim = \epsilon^\sim.$$

From (3.26) it follows that the statement "there exists a $\hat{k}_1$ such that for any $\hat{k}_2, \hat{k}_2 > \hat{k}_1$, and for any $\hat{x}^{(\hat{k}_2)}$ such that $(\hat{x}^{(\hat{k}_2)};\hat{\mathcal{D}}_{\hat{k}_2}) \in K(G_{\hat{k}_2})$ there exists a $\hat{k}_3, \hat{k}_3 < \hat{k}_1$, and an $\hat{x}^{(\hat{k}_3)}$, such that $(\hat{x}^{(k_3)};\mathcal{D}_{\hat{k}_3}) \in \hat{K}(\hat{G}_{\hat{k}_3})$ and

$$\hat{\Sigma}_{1^\sim\leq\hat{i}\leq\hat{k}_3} |\hat{x}^{(\hat{k}_3)}{}_{\hat{i}} - \hat{x}^{(\hat{k}_2)}{}_{\hat{i}}| + \hat{\Sigma}_{\hat{k}_3<\hat{i}\leq\hat{k}_2} x^{(\hat{k}_2)}{}_{\hat{i}} < \epsilon^\sim\text{"}$$

is true in $\mathcal{B}$. The statement is expressible as a sentence in the first order predicate calculus. Then it is true when re-interpreted in $\mathcal{A}$. The statement, when re-interpreted in $\mathcal{A}$, states precisely what we wish to prove.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) THE HEBREW UNIVERSITY OF JERUSALEM	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP

3. REPORT TITLE

AN APPLICATION OF NON-STANDARD ANALYSIS TO GAME THEORY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Wesley, Eugene

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1966	28	5

8a. CONTRACT OR GRANT NO. N62558-4355	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. RR 003 0701	Research Memorandum No. 20
c. Task No. NR 047-045	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics & Mathematical Statistics Branch Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

An extended (non-archimedean) real number system is used as a tool to prove certain theorems in the theory of cooperative games. The games discussed in this report involve a denumerably infinite number of players. Theorems concerning the nature of the kernel of such games are presented.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Cooperative Games						
Extended Fields of Real Numbers						
Game Theory						
Kernel of a Cooperative Game						
Logic						
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